Algebraic tori as Nisnevich sheaves with transfers

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Abstract. We relate R-equivalence on tori with Voevodsky's theory of homotopy invariant Nisnevich sheaves with transfers and effective motivic complexes.

Contents

1. Main results

Let k be a field and let T be a k-torus. The R-equivalence classes on T have been extensively studied by several authors, notably by Colliot-Thélène and Sansuc in a series of papers including [3] and [4]: they play a central rôle in many rationality issues. In this note, we show that Voevodsky's triangulated category of motives sheds a new light on this question: see Corollaries 1, 3 and 4 below.

More generally, let G be a semi-abelian variety over k, which is an extension of an abelian variety A by a torus T. Denote by HI the category of homotopy invariant Nisnevich sheaves with transfers over k in the sense of Voevodsky [19]. Then G has a natural structure of an object of HI ([17, proof of Lemma 3.2], [1, Lemma 1.3.2]). Let L be the group of cocharacters of T.

Proposition 1. *There is a natural isomorphism* $G_{-1} \xrightarrow{\sim} L$ *in* HI.

Here $_{-1}$ is the contraction operation of [18, p. 96], whose definition is recalled in the proof below.

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Proof. Recall that if $\mathcal{F}$ is a presheaf [with transfers] on smooth k-schemes, the presheaf [with transfers] $\mathcal{F}^p_{-1}$ is defined by

$$U \mapsto \operatorname{Coker}(\mathcal{F}(U \times \mathbf{A}^1) \to \mathcal{F}(U \times \mathbb{G}_m)).$$

If $\mathcal{F}$ is homotopy invariant, we may replace $U \times \mathbf{A}^1$ by U and the rational point $1 \in \mathbb{G}_m$ realises $\mathcal{F}^p_{-1}(U)$ as a functorial direct summand of $\mathcal{F}(U \times \mathbb{G}_m)$.

If $\mathcal{F}$ is a Nisnevich sheaf [with transfers], $\mathcal{F}_{-1}$ is defined as the sheaf associated to $\mathcal{F}^p_{-1}$.

Now $A(U \times \mathbf{A}^1) \xrightarrow{\sim} A(U \times \mathbb{G}_m)$ since A is an abelian variety, hence $A^p_{-1} = 0$. We therefore have an isomorphism of presheaves $T^p_{-1} \xrightarrow{\sim} G^p_{-1}$, and *a fortiori* an isomorphism of Nisnevich sheaves $T_{-1} \xrightarrow{\sim} G_{-1}$.

Let $p : \mathbb{G}_m \to \operatorname{Spec} k$ be the structural map. One easily checks that the *étale* sheaf $\operatorname{Coker}(T \xrightarrow{i} p_* p^* T)$ is canonically isomorphic to L. Since i is split, its cokernel is still L if we view it as a morphism of presheaves, hence of Nisnevich sheaves. $\square$

From now on, we assume k perfect. Let $\mathrm{DM}^{\mathrm{eff}}_-$ be the triangulated category of effective motivic complexes introduced in [19]: it has a t-structure with heart HI. It also has a tensor structure and a (partially defined) internal Hom. We then have an isomorphism

$$L[0] = G_{-1}[0] \simeq \underline{\mathrm{Hom}}_{\mathrm{DM}^{\mathrm{eff}}_-}(\mathbb{G}_m[0], G[0])$$

[10, Rk. 4.4], hence by adjunction a morphism in $\mathrm{DM}^{\mathrm{eff}}_-$

$$L[0] \otimes \mathbb{G}_m[0] \to G. \tag{1}$$

Let $\nu_{\leq 0} G[0]$ denote the cone of (1): by [11, Lemma 6.3] or [8, §2], $\nu_{\leq 0} G[0]$ is the *birational motivic complex* associated to G. We want to compute its homology sheaves.

For this, consider a coflasque resolution

$$0 \to Q \to L_0 \to L \to 0 \tag{2}$$

of L in the sense of [3, p. 179]. Taking a coflasque resolution of Q and iterating, we get a resolution of L by invertible lattices[1]:

$$\cdots \to L_n \to \cdots \to L_0 \to L \to 0. \tag{3}$$

We set

$$Q_n = \begin{cases} Q & \text{for } n = 1 \\ \operatorname{Ker}(L_{n-1} \to L_{n-2}) & \text{for } n > 1. \end{cases}$$

[1]Recall that a *lattice* is a free finitely generated Galois module; a lattice is *invertible* if it is a direct summand of a permutation lattice.

Theorem 1. *a) Let T_n denote the torus with cocharacter group L_n. Then $\nu_{\leq 0}G[0]$ is isomorphic to the complex*

$$\cdots \to T_n \to \cdots \to T_0 \to G \to 0.$$

b) Let S_n be the torus with cocharacter group Q_n. For any connected smooth k-scheme X with function field K, we have

$$H_n(\nu_{\leq 0}G[0])(X) = \begin{cases} 0 & \text{if } n < 0 \\ G(K)/R & \text{if } n = 0 \\ S_n(K)/R & \text{if } n > 0. \end{cases}$$

The proof is given in Section 2.

Corollary 1. *The assignment $Sm(k) \ni X \mapsto \bigoplus_{x \in X^{(0)}} G(k(x))/R$ provides G/R with the structure of a homotopy invariant Nisnevich sheaf with transfers. In particular, any morphism $\varphi : Y \to X$ of smooth connected k-schemes induces a morphism $\varphi^* : G(k(X))/R \to G(k(Y))/R$.* □

This functoriality is essential to formulate Theorem 2 below. For φ a closed immersion of codimension 1, it recovers a specialisation map on R-equivalence classes with respect to a discrete valuation of rank 1 which was obtained (for tori) by completely different methods, *e.g.* [4, Th. 3.1 and Cor. 4.2] or [7]. (I am indebted to Colliot-Thélène for pointing out these references.)

Corollary 2. *a) If k is finitely generated, the n-th homology sheaf of $\nu_{\leq 0}G[0]$ takes values in finitely generated abelian groups, and even in finite groups if $n > 0$ or G is a torus.*
b) If G is a torus, then $\nu_{\leq 0}G[0] = 0$ if G is split by a Galois extension E/k whose Galois group has cyclic Sylow subgroups. This condition is automatic if k is (quasi-)finite.

The proof is also given in Section 2.

Given two semi-abelian varieties G, G', we would now like to understand the maps

$$\mathrm{Hom}_k(G, G') \to \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_-}(\nu_{\leq 0}G[0], \nu_{\leq 0}G'[0]) \to \mathrm{Hom}_{\mathrm{HI}}(G/R, G'/R).$$

In Section 3, we succeed in elucidating the nature of their composition to a large extent, at least if G is a torus. Our main result, in the spirit of Yoneda's lemma, is

Theorem 2. *Let G, G' be two semi-abelian varieties, with G a torus. Suppose given, for every function field K/k, a homomorphism $f_K : G(K)/R \to G'(K)/R$ such that f_K is natural with respect to the functoriality of Corollary 1. Then*

a) There exists an extension $\tilde{G}$ of G by a permutation torus, and a homomorphism $f : \tilde{G} \to G'$ inducing (f_K).
b) f_K is surjective for all K if and only if there exist extensions $\tilde{G}, \tilde{G}'$ of G and G' by permutation tori such that f_K is induced by a split surjective homomorphism $\tilde{G} \to \tilde{G}'$.

The proof is given in §3.3. See Proposition 2, Corollary 5, Remark 4 and Proposition 3 for complements.

This relates to questions of stable birationality studied by Colliot-Thélène and Sansuc in [3] and [4], providing alternate proofs and strengthening of some of their results (at least over a perfect field). More precisely:

Corollary 3. *a) Let G' be a semi-abelian k-variety such that $G'(K)/R = 0$ for any function field K/k. Then G' is an invertible torus.*
b) In Theorem 2 b), assume that f_K is bijective for all K/k. Then there exist extensions $\tilde{G}$, $\tilde{G}'$ of G and G' by invertible tori such that f_K is induced by an isomorphism $\tilde{G} \xrightarrow{\sim} \tilde{G}'$.

Proof. a) This is the special case $G = 0$ of Theorem 2 b).

b) By Theorem 2 b), we may replace G and G' by extensions by permutation tori such that f_K is induced by a split surjection $f : G \to G'$. Let $T = \operatorname{Ker} f$. Then $T/R = 0$ universally. By a), T is invertible. □

Corollary 3 a) is a version of [4, Prop. 7.4] (taking [3, p. 199, Th. 2] into account). Theorem 2 was inspired by the desire to understand this result from a different viewpoint.

Corollary 4. *Let $f : G \dashrightarrow G'$ be a rational map of semi-abelian varieties, with G a torus. Then the following conditions are equivalent:*

(i) *$f_* : \nu_{\leq 0}G[0] \to \nu_{\leq 0}G'[0]$ is an isomorphism (see Proposition 2).*
(ii) *$f_* : G(K)/R \to G'(K)/R$ is bijective for any function field K/k.*
(iii) *f is an isomorphism, up to extensions of G and G' by invertible tori and up to a translation. (See Lemma 6.)* □

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2. Proofs of Theorem 1 and Corollary 2

Lemma 1. *The exact sequence*

$$0 \to T(k) \to G(k) \to A(k)$$

induces an exact sequence

$$0 \to T(k)/R \xrightarrow{i} G(k)/R \to A(k).$$

Proof. Let $f : \mathbf{P}^1 \dashrightarrow G$ be a k-rational map defined at 0 and 1. Its composition with the projection $G \to A$ is constant: thus the image of f lies in a T-coset of G defined by a rational point. This implies the injectivity of i, and the rest is clear. □

Let NST denote the category of Nisnevich sheaves with transfers. Recall that $\mathrm{DM}_-^{\mathrm{eff}}$ may be viewed as a localisation of $D^-(\mathrm{NST})$, and that its tensor structure is a descent of the tensor structure on the latter category [19, Prop. 3.2.3].

Lemma 2. *If G is an invertible torus, there is a canonical isomorphism in $D^-(\mathrm{NST})$*

$$L[0] \otimes \mathbb{G}_m \xrightarrow{\sim} G[0].$$

In particular, $\nu_{\leq 0} G[0] = 0$.

Proof. We reduce to the case $T = R_{E/k}\mathbb{G}_m$, where E is a finite extension of k. Let us write more precisely $\mathrm{NST}(k)$ and $\mathrm{NST}(E)$. There is a pair of adjoint functors

$$\mathrm{NST}(k) \xrightarrow{f^*} \mathrm{NST}(E), \quad \mathrm{NST}(E) \xrightarrow{f_*} \mathrm{HI}(k)$$

where $f : \operatorname{Spec} E \to \operatorname{Spec} k$ is the projection. Clearly,

$$f_*\mathbf{Z} = \mathbf{Z}_{\mathrm{tr}}(\operatorname{Spec} E), \quad f_*\mathbb{G}_m = T$$

where $\mathbf{Z}_{\mathrm{tr}}(\operatorname{Spec} E)$ is the Nisnevich sheaf with transfers represented by $\operatorname{Spec} E$. Since $\mathbf{Z}_{\mathrm{tr}}(\operatorname{Spec} E) = L$, this proves the claim. □

Proof of Theorem 1. a) Recall that L_0 is an invertible lattice chosen so that $L_0(E) \to L(E)$ is surjective for any extension E/k. In particular, (2) and (3) are exact as sequences of Nisnevich sheaves; hence $L[0]$ is isomorphic in $D^-(\mathrm{NST})$ to the complex

$$L_\cdot = \cdots \to L_n \to \cdots \to L_0 \to 0.$$

(We may view (3) as a version of Voevodsky's "canonical resolutions" as in [19, §3.2 p. 206].)

By Lemma 2, $L_n[0] \otimes \mathbb{G}_m[0] \simeq T_n[0]$ is homologically concentrated in degree 0 for all n. It follows that the complex

$$T_\cdot = \cdots \to T_n \to \cdots \to T_0 \to 0$$

is isomorphic to $L[0] \otimes \mathbb{G}_m[0]$ in $D^-(\mathrm{NST})$, hence *a fortiori* in $\mathrm{DM}_-^{\mathrm{eff}}$.

b) For any nonempty open subscheme $U \subseteq X$ we have isomorphisms

$$H_n(\nu_{\leq 0}G[0])(X) \xrightarrow{\sim} H_n(\nu_{\leq 0}G[0])(U) \xrightarrow{\sim} H_n(\nu_{\leq 0}G[0])(K) \tag{4}$$

(e.g. [8, p. 912]). By a), the right hand term is the n-th homology group of the complex

$$\cdots \to T_n(K) \to \cdots \to T_0(K) \to G(K) \to 0$$

with $G(K)$ in degree 0. By [3, p. 199, Th. 2], the sequences

$$0 \to S_1(K) \to T_0(K) \to T(K) \to T(K)/R \to 0$$
$$0 \to S_{n+1}(K) \to T_n(K) \to S_n(K) \to S_n(K)/R \to 0$$

are all exact. Using Lemma 1 for H_0, the conclusion follows from an easy diagram chase. □

Remark 1. As a corollary to Theorem 1, $S_n(K)/R$ only depends on G. This can be seen without mentioning $\mathrm{DM}_-^{\mathrm{eff}}$: in view of the reasoning just above, it suffices to construct a homotopy equivalence between two resolutions of the form (3), which easily follows from the definition of coflasque modules.

Proof of Corollary 2. a) This follows via Theorem 1 and Lemma 1 from [3, p. 200, Cor. 2] and the Mordell-Weil-Néron theorem. b) We may choose the L_n, hence the S_n split by E/k. The conclusion now follows from Theorem 1 and [3, p. 200, Cor. 3]. The last claim is clear. □

Remark 2. In characteristic $p > 0$, all finitely generated perfect fields are finite. To give some contents to Corollary 2 a) in this characteristic, one may pass to the perfect [one should say radicial] closure k of a finitely generated field k_0. If G is a semi-abelian k-variety, it is defined over some finite extension k_1 of k_0. If k_2/k_1 is a finite (purely inseparable) subextension of k/k_1, then the composition

$$G(k_2) \xrightarrow{N_{k_2/k_1}} G(k_1) \to G(k_2)$$

equals multiplication by $[k_2 : k_1]$. Hence Corollary 2 a) remains true at least after inverting p.

3. Stable birationality

If X is a smooth variety over a field k, we write $\mathrm{Alb}(X)$ for its generalised Albanese variety in the sense of Serre [16]: it is a semi-abelian variety, and a rational point $x_0 \in X$ determines a morphism $X \to \mathrm{Alb}(X)$ which is universal for morphisms from X to semi-abelian varieties sending x_0 to 0.

We also write $\mathrm{NS}(X)$ for the group of cycles of codimension 1 on X modulo algebraic equivalence. This group is finitely generated if k is algebraically closed [9, Th. 3].

3.1. **Well-known lemmas.** I include proofs for lack of reference.

Lemma 3. *a) Let G, G' be two semi-abelian k-varieties. Then any k-morphism $f : G \to G'$ can be written uniquely $f = f(0) + f'$, where f' is a homomorphism.*
b) For any semi-abelian k-variety G, the canonical map $G \to \mathrm{Alb}(G)$ sending 0 to 0 is an isomorphism.

Proof. a) amounts to showing that if $f(0) = 0$, then f is a homomorphism. By an adjunction game, this is equivalent to b). Let us give two proofs: one of a) and one of b).

Proof of a). We may assume k to be a universal domain. The staement is classical for abelian varieties [15, p. 41, Cor. 1] and an easy computation for tori. In the general case, let T, T' be the toric parts of G and G' and A, A' be their abelian parts. Let $g \in G(k)$. As any morphism from T to A' is constant, the k-morphism

$$\varphi_g : T \ni t \mapsto f(g+t) - f(g) \in G'$$

(which sends 0 to 0) lands in T', hence is a homomorphism. Therefore it only depends on the image of g in $A(k)$. This defines a morphism $\varphi : A \to \underline{\mathrm{Hom}}(T, T')$, which must be constant with value $\varphi_0 = f$. It follows that

$$(g, h) \mapsto f(g+h) - f(g) - f(h)$$

induces a morphism $A \times A \to T'$. Such a morphism is constant, of value 0.

Proof of b). This is true if G is abelian, by rigidity and the equivalence between a) and b). In general, any morphism from G to an abelian variety is trivial on T. This shows that the abelian part of $\mathrm{Alb}(G)$ is A. Let $T' = \mathrm{Ker}(\mathrm{Alb}(G) \to A)$. We also have the counit morphism $\mathrm{Alb}(G) \to G$, and the composition $G \to \mathrm{Alb}(G) \to G$ is the identity. Thus T is a direct summand of T'. It suffices to show that $\dim T' = \dim T$. Going to the algebraic closure, we may reduce to $T = \mathbb{G}_m$.

Then consider the line bundle completion $\bar{G} \to A$ of the $\mathbb{G}_m$-bundle $G \to A$. It is sufficient to show that the kernel of

$$\mathrm{Alb}(G) \to \mathrm{Alb}(\bar{G}) = A$$

is 1-dimensional. This follows for example from [1, Cor. 10.5.1]. □

Lemma 4. *Suppose k algebraically closed, and let G be a semi-abelian k-variety. Let A be the abelian quotient of G. Then the map*

$$\mathrm{NS}(A) \to \mathrm{NS}(G) \tag{5}$$

is an isomorphism.

Proof. Let $T = \mathrm{Ker}(G \to A)$ and $X(T)$ be its character group. Choosing a basis (e_i) of $X(T)$, we may complete the $\mathbb{G}_m^n$-torsor G into a product of line bundles $\bar{G} \to A$. The surjection

$$\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{Pic}(\bar{G}) \twoheadrightarrow \mathrm{Pic}(G)$$

show the surjectivity of (5). Its kernel is generated by the classes of the irreducible components D_i of the divisor with normal crossings $\bar{G} - G$. These components correspond to the basis elements e_i. Since the corresponding $\mathbb{G}_m$-bundle is a group extension of A by $\mathbb{G}_m$, the class of the 0 section of its line bundle completion lies in $\mathrm{Pic}^0(A)$, hence goes to 0 in $\mathrm{NS}(\bar{G})$. □

Lemma 5. *Let X be a smooth k-variety, and let $U \subseteq X$ be a dense open subset. Then there is an exact sequence of semi-abelian varieties*

$$0 \to T \to \mathrm{Alb}(U) \to \mathrm{Alb}(X) \to 0$$

with T a torus. If $\mathrm{NS}(\bar{U}) = 0$ (this happens if U is small enough), there is an exact sequence of character groups

$$0 \to X(T) \to \bigoplus_{x \in X^{(1)} - U^{(1)}} \mathbf{Z} \to \mathrm{NS}(\bar{X}) \to 0.$$

Proof. This follows for example from [1, Cor. 10.5.1]. □

Lemma 6. *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian k-varieties, with G a torus. Then there exists an extension $\tilde{G}$ of G by a permutation torus and a homomorphism $\tilde{f} : \tilde{G} \to G'$ which extends f up to translation in the following sense: there exists a rational section $s : G \dashrightarrow \tilde{G}$ of the projection $\pi : \tilde{G} \to G$ and a rational point $g' \in G'(k)$ such that $f = \tilde{f}s + g'$. If f is defined at 0_G and sends it to $0_{G'}$, then $g' = 0$.*

Proof. Let U be an open subset of G where f is defined. We define $\tilde{G} = \mathrm{Alb}(U)$. Applying Lemmas 5 and 3 b) and using $\mathrm{NS}(\bar{G}) = 0$, we get an extension

$$0 \to P \to \tilde{G} \to G \to 0$$

where P is a permutation torus, as well as a morphism $\tilde{f} = \mathrm{Alb}(f) : \tilde{G} \to G'$.

Let us first assume k infinite. Then $U(k) \neq \emptyset$ because G is unirational. A rational point $g \in U$ defines an Albanese map $s : U \to \tilde{G}$ sending g to $0_{\tilde{G}}$. Since P is a permutation torus, $g \in G(k)$ lifts to $\tilde{g} \in \tilde{G}(k)$ (Hilbert 90) and we may replace s by a morphism sending g to $\tilde{g}$. Then s is a rational section of π. Moreover, $f = \tilde{f}s + g'$ with $g' = f(g) - \tilde{f}(\tilde{g})$. The last assertion follows.

If k is finite, then U has at least a zero-cycle g of degree 1, which is enough to define the Albanese map s. We then proceed as above (lift every closed point involved in g to a closed point of $\tilde{G}$ with the same residue field). □

Lemma 7. *Let G be a finite group, and let A be a finitely generated G-module. Then*
a) There exists a short exact sequence of G-modules $0 \to P \to F \to A \to 0$, with F torsion-free and flasque, and P permutation.
b) Let B be another finitely generated G-module, and let $0 \to P' \to E \to B \to 0$ be an exact sequence with P' an invertible module. Then any G-morphism $f : A \to B$ lifts to $\tilde{f} : F \to E$.

Proof. a) is the contents of [4, Lemma 0.6, (0.6.2)]. b) The obstruction to lifting f lies in $\mathrm{Ext}^1_G(F, P') = 0$ [3, p. 182, Lemme 9]. □

3.2. **Functoriality of $\nu_{\leq 0}G$.** We now assume k perfect.

Lemma 8. *Let*

$$0 \to P \to G \to H \to 0 \tag{6}$$

be an exact sequence of semi-abelian varieties, with P an invertible torus. Then $\nu_{\leq 0}G[0] \xrightarrow{\sim} \nu_{\leq 0}H[0]$.

Proof. As P is invertible, (6) is exact in NST hence defines an exact triangle

$$P[0] \to G[0] \to H[0] \xrightarrow{+1}$$

in $\mathrm{DM}^{\mathrm{eff}}_-$. The conclusion then follows from Lemma 2. □

Proposition 2. *Let G, G' be two semi-abelian k-varieties, with G a torus. Then a rational map $f : G \dashrightarrow G'$ induces a morphism $f_* : \nu_{\leq 0}G[0] \to \nu_{\leq 0}G'[0]$, hence a homomorphism $f_* : G(K)/R \to G'(K)/R$ for any extension K/k. If K is infinite, f_* agrees up to translation with the morphism induced by f via the isomorphism $U(K)/R \xrightarrow{\sim} G(K)/R$ from* [3, p. 196 Prop. 11], *where U is an open subset of definition of f.*

Proof. By Lemma 6, f induces a homomorphism $\tilde{G} \to G'$ where $\tilde{G}$ is an extension of G by a permutation torus. By Lemma 8, the induced

morphism
$$\nu_{\le 0}\tilde{G}[0] \to \nu_{\le 0}G'[0]$$
factors through a morphism $f_* : \nu_{\le 0}G[0] \to \nu_{\le 0}G'[0]$.

The claims about R-equivalence classes follow from Theorem 1 b) and Lemma 6. □

Remark 3. The proof shows that $f'_* = f_*$ if f' differs from f by a translation by an element of $G(k)$ or $G'(k)$.

Corollary 5. *If T and T' are birationally equivalent k-tori, then $\nu_{\le 0}T[0] \simeq \nu_{\le 0}T'[0]$. In particular, the groups $T(k)/R$ and $T'(k)/R$ are isomorphic.*

Proof. The proof of Proposition 2 shows that $f \mapsto f_*$ is functorial for composable rational maps between tori. Let $f : T \dashrightarrow T'$ be a birational isomorphism, and let $g : T' \dashrightarrow T$ be the inverse birational isomorphism. Then we have $g_* f_* = 1_{\nu_{\le 0}T[0]}$ and $f_* g_* = 1_{\nu_{\le 0}T'[0]}$. The last claim follows from Theorem 1. □

Remark 4. It is proven in [3] that a birational isomorphism of tori $f : T \dashrightarrow T'$ induces a set-theoretic bijection $f_* : T(k)/R \xrightarrow{\sim} T'(k)/R$ (p. 197, Cor. to Prop. 11) and that the *group* $T(k)/R$ is abstractly a birational invariant of T (p. 200, Cor. 4). The proof above shows that f_* is an isomorphism of groups if f respects the origins of T and T'. This solves the question raised in [3, mid. p. 397]. The proofs of Lemma 6 and Proposition 2 may be seen as dual to the proof of [3, p. 189, Prop. 5], and are directly inspired from it.

3.3. **Faithfulness and fullness.**

Proposition 3. *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian varieties, with G a torus. Assume that the map $f_* : G(K)/R \to G'(K)/R$ from Proposition 2 is identically 0 when K runs through the finitely generated extensions of k. Then there exists a permutation torus P and a factorisation of f as*
$$G \overset{\tilde{f}}{\dashrightarrow} P \xrightarrow{g} G'$$
where $\tilde{f}$ is a rational map and g is a homomorphism. If f is a morphism, we may choose $\tilde{f}$ as a homomorphism.
Conversely, if there is such a factorisation, then $f_* : \nu_{\le 0}G[0] \to \nu_{\le 0}G'[0]$ is the 0 morphism.

Proof. By Lemma 6, we may reduce to the case where f is a morphism. Let $K = k(G)$. By hypothesis, the image of the generic point $\eta_G \in G(K)$ is R-equivalent to 0 on $G'(K)$. By a lemma of Gille [6, Lemme

II.1.1 b)], it is directly R-equivalent to 0: in other words, there exists a rational map $h : G \times \mathbf{A}^1 \dashrightarrow G'$, defined in the neighbourhood of 0 and 1, such that $h_{|G\times\{0\}} = 0$ and $h_{|G\times\{1\}} = f$.

Let $U \subseteq G \times \mathbf{A}^1$ be an open set of definition of h. The 0 and 1-sections of $G \times \mathbf{A}^1 \to G$ induce sections

$$s_0, s_1 : G \to \mathrm{Alb}(U)$$

of the projection $\pi : \mathrm{Alb}(U) \to \mathrm{Alb}(G \times \mathbf{A}^1) = G$ such that $\mathrm{Alb}(h) \circ s_0 = 0$ and $\mathrm{Alb}(h) \circ s_1 = f$. If $P = \operatorname{Ker}\pi$, then $s_0 - s_1$ induces a homomorphism $\tilde{f} : G \to P$ such that the composition

$$G \xrightarrow{\tilde{f}} P \to \mathrm{Alb}(U) \xrightarrow{\mathrm{Alb}(h)} G'$$

equals f. Finally, P is a permutation torus by Lemma 5.

The last claim follows from Lemma 2. □

Proof of Theorem 2. a) Take $K = k(G)$. The image of the generic point η_G by f_K lifts to a (non unique) rational map $f : G \dashrightarrow G'$. Using Lemma 6, we may extend f to a homomorphism

$$\tilde{f} : \tilde{G} \to G'$$

where $\tilde{G}$ is an extension of G by a permutation torus P. Since $\tilde{G}(K)/R \xrightarrow{\sim} G(K)/R$, we reduce to $\tilde{G} = G$ and $\tilde{f} = f$.

Let L/k be a fonction field, and let $g \in G(L)$. Then g arises from a morphism $g : X \to G$ for a suitable smooth model X of L. By assumption on $K \mapsto f_K$, the diagram

$$\begin{array}{ccc} G(K)/R & \xrightarrow{f_K} & G'(K)/R \\ {\scriptstyle g^*}\big\downarrow & & {\scriptstyle g^*}\big\downarrow \\ G(L)/R & \xrightarrow{f_L} & G'(L)/R \end{array}$$

commutes. Applying this to $\eta_K \in G(K)$, we find that $f_L([g]) = [g \circ f]$, which means that f_L is the map induced by f.

b) The hypothesis implies that $G'(E)/R = 0$ for any algebraically closed extension E/k, which in turn implies that G' is also a torus. Applying a), we may, and do, convert f into a true homomorphism by replacing G by a suitable extension by a permutation torus. Applying Lemma 7 a) to the cocharacter group of G, we get a resolution $0 \to P_1 \to Q \to G \to 0$ with Q coflasque and P_1 permutation. Hence we may (and do) further assume G coflasque.

Let $K = k(G')$ and choose some $g \in G(K)$ mapping modulo R-equivalence to the generic point of G'. Then g defines a rational map

$g : G' \dashrightarrow G$ such that fg is R-equivalent to $1_{G'}$. It follows that the induced map

$$1 - fg : G'/R \to G'/R \tag{7}$$

is identically 0.

Reapplying Lemma 6, we may find an extension $\tilde{G}'$ of G' by a suitable permutation torus which converts g into a true homomorphism. Since G is coflasque, Lemma 7 b) shows that $f : G \to G'$ lifts to $\tilde{f} : G \to \tilde{G}'$. Then (7) is still identically 0 when replacing (G', f) by $(\tilde{G}', \tilde{f})$.

Summarising: we have replaced the initial G and G' by suitable extensions by permutation tori, such that f lifts to these extensions and there is a homomorphism $g : G' \to G$ such that (7) vanishes identically. Hence $1-fg$ factors through a permutation torus P thanks to Proposition 3. Write $u : G' \to P$ and $v : P \to G'$ for homomorphisms such that $1 - fg = vu$. Let $G_1 = G \times P$ and consider the maps

$$f_1 = (f, v) : G_1 \to G', \qquad g_1 = \begin{pmatrix} g \\ u \end{pmatrix} : G' \to G_1.$$

Then $f_1 g_1 = 1$ and G' is a direct summand of G_1 as requested. □

4. Some open questions

Question 1. Are lemma 6 and Proposition 2 still true when G is not a torus?

This is far from clear in general, starting with the case where G is an abelian variety and G' a torus. Let me give a positive answer in the case of an elliptic curve.

Proposition 4. *The answer to Question 1 is yes if the abelian part A of G is an elliptic curve.*

Proof. Arguing as in the proof of Proposition 2, we get for an open subset $U \subseteq G$ of definition for f an exact sequence

$$0 \to \mathbb{G}_m \to P \to \mathrm{Alb}(U) \to G \to 0$$

where P is a permutation torus. Here we used that $\mathrm{NS}(\bar{G}) \simeq \mathbf{Z}$, which follows from Lemma 4.

The character group $X(P)$ has as a basis the geometric irreducible components of codimension 1 of $G - U$. Up to shrinking U, we may assume that $G - U$ contains the inverse image D of $0 \in A$. As the divisor class of 0 generates $\mathrm{NS}(\bar{A})$, D provides a Galois-equivariant splitting of the map $\mathbb{G}_m \to P$. Thus its cokernel is still a permutation torus, and we conclude as before. □

Question 2. Can one formulate a version of Theorem 2 and Corollary 3 providing a description of the groups $\operatorname{Hom}_{\mathrm{DM}_-^{\mathrm{eff}}}(\nu_{\le 0}G[0], \nu_{\le 0}G'[0])$ and $\operatorname{Hom}_{\mathrm{HI}}(G/R, G'/R)$ (at least when G and G' are tori)?

The proof of Theorem 2 suggests the presence of a closed model structure on the category of tori (or lattices), which might provide an answer to this question.

For the last question, let G be a semi-abelian variety. Forgetting its group structure, it has a motive $M(G) \in \mathrm{DM}_-^{\mathrm{eff}}$. Recall the canonical morphism

$$M(G) \to G[0]$$

induced by the "sum" maps

$$c(X, G) \xrightarrow{\sigma} G(X) \tag{8}$$

for smooth varieties X ([17, (6), (7)], [1, §1.3]).

The morphism (8) has a canonical section

$$G(X) \xrightarrow{\gamma} c(X, G) \tag{9}$$

given by the graph of a morphism: this section is functorial in X but is not additive.

Consider now a smooth equivariant compactification $\bar{G}$ of G. It exists in all characteristics. For tori, this is written up in [2]. The general case reduces to this one by the following elegant argument I learned from M. Brion: if G is an extension of an abelian variety A by a torus T, take a smooth projective equivariant compactification Y of T. Then the bundle $G \times^T Y$ associated to the T-torsor $G \to A$ also exists: this is the desired compactification.

Then we have a diagram of birational motives

$$\begin{array}{ccc} \nu_{\le 0}M(G) & \xrightarrow{\sim} & \nu_{\le 0}M(\bar{G}) \\ {\scriptstyle \nu_{\le 0}\sigma}\downarrow & & \\ \nu_{\le 0}G[0]. & & \end{array} \tag{10}$$

By [11], we have $H_0(\nu_{\le 0}M(\bar{G}))(X) = CH_0(\bar{G}_{k(X)})$ for any smooth connected X. Hence the above diagram induces a homomorphism

$$CH_0(\bar{G}_{k(X)}) \to G(k(X))/R \tag{11}$$

which is natural in X for the action of finite correspondences (compare Corollary 1). One can probably check that this is the homomorphism of [12, (17) p. 78], reformulating [3, Proposition 12 p. 198]. Similarly, the set-theoretic map

$$G(k(X))/R \to CH_0(\bar{G}_{k(X)}) \tag{12}$$

of [3, p. 197] can presumably be recovered as a birational version of (9), using perhaps the homotopy category of schemes of Morel and Voevodsky [14].

In [12], Merkurjev shows that (11) is an isomorphism for G a torus of dimension at most 3. This suggests:

Question 3. Is the map $\nu_{\le 0}\sigma$ of Diagram (10) an isomorphism when G is a torus of dimension ≤ 3?

In [13], Merkurjev gives examples of tori G for which (12) is not a homomorphism; hence its (additive) left inverse (11) cannot be an isomorphism. Merkurjev's examples are of the form $G = R^1_{K/k}\mathbb{G}_m \times R^1_{L/k}\mathbb{G}_m$, where K and L are distinct biquadratic extensions of k. This suggests:

Question 4. Can one study Merkurjev's examples from the above viewpoint? More generally, what is the nature of the map $\nu_{\le 0}\sigma$ of Diagram (10)?

We leave all these questions to the interested reader.

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